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Black holes, stability and geodesics in modified gravity theories

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ACADEMIC DISSERTATION

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We must not forget that when radium was discovered no one knew that it would prove useful in hospitals. The work was one of pure science. And this is a proof that scientific work must not be considered from the point of view of the direct usefulness of it. It must be done for itself, for the beauty of science, and then there is always the chance that a scientific discovery may become like the radium a benefit for humanity.

–Maria Skłodowska-Curie

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Abstract

For a hundred years Einstein's general relativity (GR) has persisted as the standard model of gravity. To date, no observations conflict it. Moreover, GR has predicted effects that have been later confirmed, such as the deflection of light, the redshift of light in gravitational field, and gravitational waves.

Modifications to GR have been studied since the early days of relativistic gravity. In this thesis, we present three projects on different modified gravity models. These include Palatini f(R), bimetric variational principle, and scalar-tensor theories.

In Palatini f(R) theories, a function of the Ricci scalar, f(R), acts as the gravitational Lagrangian. The connection is independent of the metric. In GR, the function $f(R) = R$, and the Einstein equations follow regardless of whether or not the connection is independent.

We show that for a system of compact objects, the difference between Palatini f(R) and GR is the scaling of masses. However, without complementary measurement of masses, such systems are observationally indistinguishable.

In bimetric variational principle, the independent spacetime connection is constructed of a tensor apart from the physical metric. The physical metric and this new tensor then give the field equations. We study Einstein-Hilbert and f(R) actions of the Ricci scalar of the connection. We use ADM formalism to show that without further constraints the resulting Hamiltonian contains a dynamical variable without a lower bound. This leads to decaying to lower energy states by radiating energy ad infinitum. Hence, these theories are physically unviable.

We also study scalar-tensor theories with disformally coupled matter. In these theories matter couples to the scalar field and its derivatives.

We focus on a system of disformally coupled matter surrounding a black hole. In Brans-Dicke-type theories such systems suffer from the so called spontaneous scalarisation. In it, the scalar field develops stable scalar hair around the black hole. This conflicts the no-hair theorem according to which the only observable properties of a black hole are its mass, angular momentum, and electric charge. We show that disformal coupling can further destabilise the system. We find a range of disformal coupling strengths where the coupling enhances the scalarisation. Outside this range, the effect of the disformal coupling is stabilising.

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Helsinki, August 2017

Hannu Nyrhinen

List of Included Papers

The three articles [1–3] included in this thesis are:

1. Enqvist K., Koivisto T., and Nyrhinen H. J. *Binary systems in Palatini-f(R) gravity*. Phys. Rev. D88(10), 104008, 2013. 1308.0988

2. Golovnev A., Karciauskas M., and Nyrhinen H. J. *ADM analysis of gravity models within the framework of bimetric variational formalism*. JCAP. 1505(05), 021, 2015. 1412.0637

3. Koivisto T. and Nyrhinen H. J. *Stability of disformally coupled accretion disks*. 2015. 1503.02063

Authors are listed alphabetically according to high energy physics convention.

Author's contribution

Paper 1: In this article we calculate the post-Newtonian expansion for systems consisting of compact objects in Palatini f(R) gravity. The author did the calculations following the DIRE method described for general relativity in [4–6].

Paper 2: In this article we studied a class of theories where two rank two tenors give the structure of the spacetime. One of these tensors is the physical metric and the other gives the independent connection. The author calculated the Hamiltonian from an initially promising Lagrangian density.

Paper 3: In this article we studied matter surrounding a black hole. We considered theories in which a scalar field couples to matter via disformal coupling. The author participated in analysing the initial system consisting of a Schwarzschild black hole and expanded the analysis to Kerr black holes.

The author also participated in writing of each article.

Contents

Chapter 1

Introduction and a short history of relativistic gravity

Of the four fundamental interactions gravity is by far the weakest. Yet it dictates the structure and the evolution of the Universe. It is also the interaction for which the widely accepted model is the oldest. Formulated a hundred years ago, general relativity (GR) [7] is still the simplest measurement-wise accurate theory of gravity. However, gravity is also to date the sole interaction that we cannot properly unify with the rest. Moreover, although GR agrees with observations, there still remains room for speculation on the nature of the gravitational interaction: We are not sure even of the number of gravitational fields, and much less so of their mathematical formulation. This thesis is based on three articles, all looking at different possible modifications outside GR.

Partly due to the weakness of gravity, measuring the relativistic corrections to Newtonian gravitational force is anything but a trivial task. Einstein suggested to look for the bending of light rays from distant objects, the redshift of electromagnetic radiation (light), and gravitational waves emitted by massive binary systems. The last of these three was detected in 2015 by the LIGO experiment, almost exactly one hundred years after the theory was first published [8].

Detecting deviations from GR means being able to detect effects that are second-order corrections to Newton's force and hence even harder to measure. This of course does not hinder the creation and theoretical research of models of gravity, but until we find GR grossly in conflict with observations, it is more or less up to anyone's preferences where and what to look for.

In this thesis we will go through some of the aspects that are common for modified gravity theories. We will then focus especially on a few particular phenomena arising within some of the models.

1.1 Historical Background

Gravity research has been at the core of the study of the physical world at least since the era of enlightenment. The stories of Newton getting hit by an apple falling off a tree and Galilei throwing things from the Leaning Tower of Pisa are well known.[1] Whether or not either of these stories is historically accurate is irrelevant, but their teaching is still valid. The perhaps counterintuitive ideas that all massive objects attract one another at equally strong force and that heavier objects should fall at the same velocity as lighter ones is an example of what is nowadays known as the equivalence principle. As tests and our understanding of nature has deepened, this principle also has evolved. In section 2.3 we introduce three formulations of equivalence principles in more detail, the weak, the strong, and the Einstein equivalence principle. To date, their validity remains a topic of experimental research. So far no conflicts have been detected. On the other hand, violations of the equivalence principles have not been ruled out either. This enables dropping the assumption of strong equivalence leading to models that are typically called modified gravity.[2]

The requirement of equivalence of inertial observers lead to the formulation of special relativity. The Maxwell equations imply a wave equation for electromagnetic radiation. The equation describes a wave that moves with the same constant velocity, the speed of light, irrespective of the movement of the observer. Moreover, the wave equation requires Lorentzian instead of the Galilean transformation in order to stay covariant between frames. The independence of the chosen inertial frame is one of the requirements of the strong equivalence principle.

Another issue to solve was the concept of inertial frames. If observers in all inertial reference frames are equivalent, then we need to know which frames are inertial. Einstein suggested that freely falling observers should also be included [11]. "A freely falling man does not feel his weight." That is, there is no difference in performing experiments floating in free space or in free-fall. This was of course not the case in Newtonian mechanics where the rest frames of accelerated observers are not inertial. These considerations lead to the formulation of general relativity in 1915 [7, 11].

General relativity was not the first attempt at a relativistic theory of gravity. The article containing the final equations of motion for the gravitational field was the fourth Einstein submitted on the subject only that month [12]. The problem was to formulate covariantly constant equations that would give the correct precession for the perihelion shift of the planet Mercury. Almost at the same time Hilbert came up with the Lagrangian that produces the same equations of motion [13].

The process was anything but straight forward and other suggestions for theories also existed. For example, in 1913 a Finnish physicist Gunnar Nordström published a scalar theory. There the field

[1] Apparently Newton's version of the anecdote was that he saw an apple fall [9].

[2] We will adopt a classification where a model is categorised as modified gravity if it violates the strong equivalence principle after [10].

equations read [14, 15]

$$\phi\,\Box\phi = -\nu T \tag{1.1}$$

or

$$R = \kappa T. \tag{1.2}$$

Here ν and κ are constants whose values are not important for this discussion.

Nordström's theory is successful in that it satisfies the strong equivalence principle. However, unlike GR it does not predict the bending of light. In 1919 the deflection of starlight near the Sun was measured and the lightbending thus confirmed [16]. Therefore, Nordström's scalar theory can not be considered a viable theory of gravity. Instead, the bending observation was considered a "triumph" for GR [17].

The major philosophical shift that relativity brought along was to promote time as the fourth spacetime coordinate instead of being a mere parameter. Not only that, but sometimes the distinction between timelike and spacelike natures of different coordinates is less than obvious. Einstein emphasised this by naming the four coordinates as (x^1, x^2, x^3, x^4) rather that the usual (x, y, z, t).

However, we must admit that cosmologically, as well as from purely everyday-life point of view, time is fundamentally different from the other three dimensions. The basic assumption of modern cosmology, the cosmological principle, states that the Universe is spatially homogeneous and isotropic. This assumption seems to hold experimentally the better the larger the scales being observed. Nonetheless, it seems it does not hold for the fourth direction. Our universe is expanding at an accelerated rate and its structure, density, temperature, and energy content have all been totally different in the past. Likewise to the best of our knowledge, they will be quite different in the future. In short, unlike spatial dimensions, time respects neither homogeneity nor isotropy.

After the observation of the bending of starlight near the Sun, general relativity was largely accepted as the theory of gravity that best describes nature. However, it faded somewhat into the background of physics research due to its heavy mathematical machinery. In addition, there were no objects apart from the Sun and the planet Mercury in which relativistic effects would be measurable. This changed beginning in the 1960's with new experiments such as measuring the gravitational shift of the frequency of light, and astronomical observations such as quasars, pulsars, and cosmic background radiation [18].

General relativity is not the only relativistic theory of gravity, and the research on gravity did not stop in the 1920's. By the sixties some mathematical work had been performed, which is relevant for this thesis; we will especially focus on the so called Palatini formulation as well as considerations on bimetric theories [19–21]. After that, during the past fifty or so years there has been rapid development in gravity research, both experimental and theoretical. The development as well as the number of research papers on gravity theories has grown acceleratingly after the discovery of accelerating expansion rate of the Universe. We will go through some of this development in later chapters.

1.2 Experimental Measurement

Over the years there has been much work to test the accuracy of general relativity and modified gravity theories. This includes observations related to solar system, to pulsars, and to other more distant compact objects [18, 22–25]. The standard model of cosmology, the so called ΛCDM model, is tested on cosmological observations [26–28]. Often these tests assume that gravity is weak, and also that relative velocities of the physical objects are slow compared to the speed of light. The recent direct detection of gravitational waves produced by a collision and merger of two black holes is remarkable in that it gives the first glimpse to phenomena in the strong gravity regime. The weak field approximation does not hold all the way from early inspiral to ringdown phase. Therefore, we must model the system numerically in order to test the theory all the way through the merger.

The gravitational field equations are a set of coupled non-linear partial differential equations. Therefore typically, solving them analytically or even numerically is very difficult, and computationally challenging. The weak field approximation enables lighter computations and easier solving of the equations. One of these approximation methods is used in our article [1]. In section 2.2.2 we present another one, the so called Parametrised Post-Newtonian (PPN) expansion. These expansion methods make the equations long instead of difficult. This also leads to computational difficulties when increased accuracy is required. However, the upside is that this allows us to compare the theories to observations in weak field environments.

So far testing the weak field has been enough: in figure 1.1 the predicted change of the period of a binary pulsar system is compared with observations. Over an observation period of thirty years all the observations agree with general relativistic prediction to measurable accuracy. So far, none of the other tests has been able to find significant deviation from GR either.

Without observations we cannot be sure of the accuracy of theories in the strong field regime. In principle new phenomena may arise in stronger fields, for instance due to shorter range interaction, rapid velocities or for whatever unforeseen reason. On the other hand, in many models modifications that are significant on large scales also become important on smaller scales or in the strong gravity environments. This allows us to constrain or even rule out some of those models.

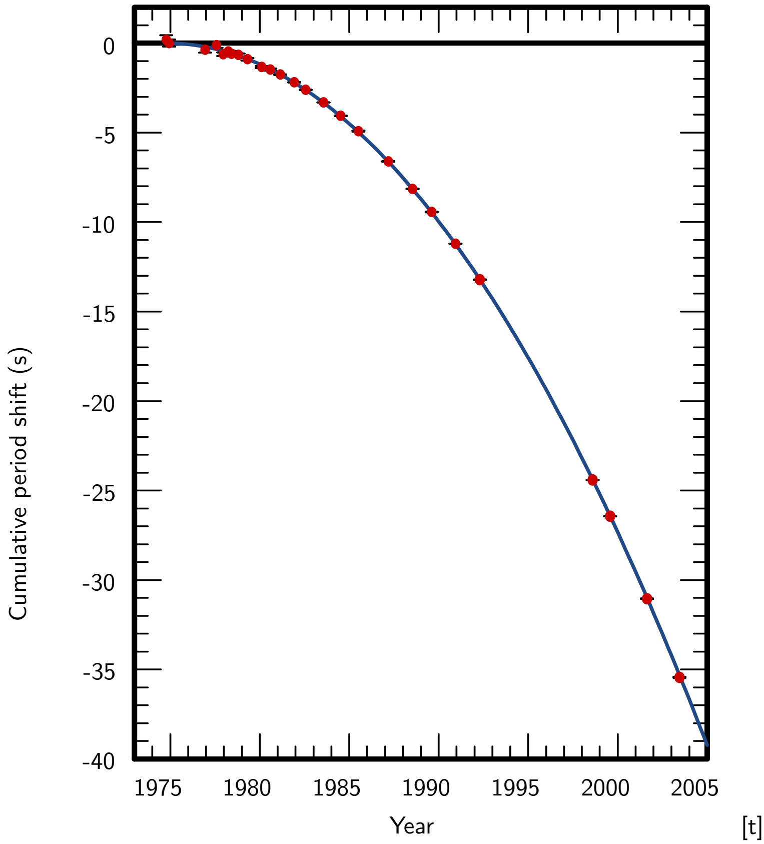
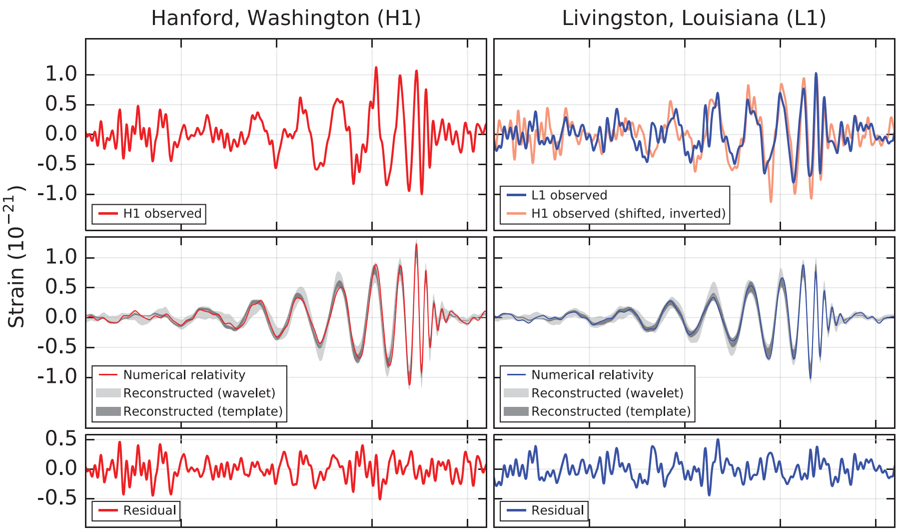
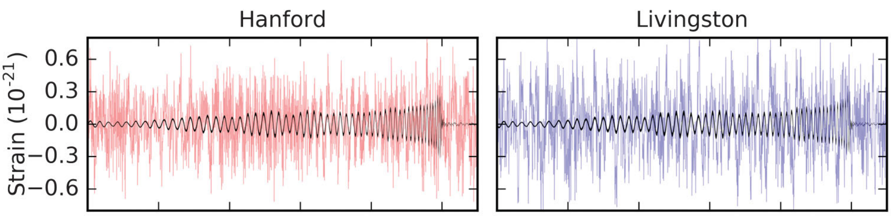


Figure 1.1: Cumulative shift in the periastron period in seconds for the binary star system PSR B1913+16 [The Hulse-Taylor binary] as the system loses energy by gravitational wave emission. Red points are experimental data, and the blue [solid] line is the shift predicted by general relativity. Error bars are shown, but are generally too small to be seen as distinct for most points. Data from [23], picture and caption from [29].



(a) Detection of GW150914. *Top row:* Detector strain from the two sites, *Middle row:* Numerical relativity model based on parameters that are consistent with GW150914 (solid line), binary black hole merger template waveform (dark grey) and reconstructed strain signal as a linear combination of sine-Gaussian wavelets (light grey) and *Bottom line:* Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. [8]



(b) Detection of GW151226. Strain data from the two detectors. Also shown (black) is the best-match template from a nonprecessing spin waveform model reconstructed using a Bayesian analysis. The thickness of the line indicates the 90 % credible region. [30]

Figure 1.2: The detection of the two binary mergers. The first detection, GW150914, shows a merger of two supposedly black holes of masses $m_{1,2} \sim 30M_{\odot}$. The second, GW151226, was a merger of binary system of masses $m_{1,2} \sim 10M_{\odot}$. Due to the masses of the binaries GW150914 was detectable for a fewer waves but had larger amplitude whereas GW151226 was detectable for longer time but with smaller amplitude. Hence, the latter wave gives tighter constrains for post-Newtonian parameters while the former tells more about the final merger-ringdown phase.

1.2.1 About LIGO

The direct detection of gravity waves [8] was the first glimpse into strong gravity phenomena. The waves GW150914 and GW151226 were formed in a collision and merger of two black holes. Modelling such events force us to relax our assumptions of weak fields and slow velocities. In figure 1.2 we present the signals from the two events caught in 2015.

The detection also represents the first time we have made astronomical observations using gravitational instead of electromagnetic (EM) signals or neutrinos. This is remarkable from the point of view of observations in that black holes are first and foremost gravitational objects: It is not impossible for them to carry electric charge or magnetic field but as they generally do not emit EM radiation, it is not possible to observe them directly through EM telescopes. So, in a way this was also the first direct observation of black holes. To be more precise: these were the first observations of binary black hole mergers, as single black holes do not emit gravitational radiation by themselves.

LIGO experiment consists of two L-shaped laser detectors [8]. The detectors observe the strain of the two arms via changes in the interference of laser beams travelling in each arm. That is, they measure changes in the arms' lengths caused by the passing gravitational waves.

To be able to separate strain by gravitational waves from local sources of vibration two "widely separated" detectors are used to verify the observations [8]. The LIGO detectors are positioned 3002 kilometres apart in Livingston, Louisiana and in Hanford Site near Richland, Washington, in the United States of America.

A gravitational wave travelling through the Earth will leave similar signals in the data of the two detectors. The orientation of the two L's causes slightly different signal strengths between the detectors. This, added to the slight time delay between the signals, allows one to narrow down the direction in which the event took place. When in future more detectors are added to the network, the directional precision will increase, and also polarisation measurements become possible.

The first of the two waves, GW150914 [8], was emitted by a merging binary black hole. The masses of the objects were of the order $30M_{\odot}$, which made the amplitude large enough to be visible in the strain signal (see figure 1.2a). The downside of the large size is that LIGO was only able to detect less than ten final revolutions of the inspiral.

The second wave, GW151226 [30], was different in that it was frequencywise more suited for LIGO. In this merger the masses were of the order $10M_{\odot}$. Hence, its amplitude was lower but the wave could be observed longer (see figure 1.2b). Due to longer observation time, this latter detection can be used to constrain post-Newtonian parameters that describe the earlier inspiral phase of the binary system [31]. We will go through these parameters and the post-Newtonian expansion later in this thesis and in [1].

The wave parameters are found from the signal using simulated data and templates of such wavesignals. In GW150914 LIGO team also constructed a corresponding signal as a linear combination of (sine-Gaussian) waves. Best fit parameters are then extracted from these best match models. The result from both events are compatible with general relativistic models of two merging black holes.

However, for this thesis the most interesting question is what these events can tell about alternative theories of gravity. As [31] points out we can constrain post-Newtonian parameters using the inspiral signal. However, we do not have sufficient model templates for modified gravity theories for the merger-ringdown phase. Hence, more work is needed to properly compare the strong gravity phenomena in modified gravity to signals from these final phases.

1.2.2 Future experiments

Future experiments will probe both strong gravity on the small scales and dark energy or modified gravity on cosmological scales. Gravitational wave astronomy will become a significant channel in observing the sky: LIGO will be accompanied by ground based advanced Virgo, aLIGO detector in India and KAGRA in Japan as well as satellite mission LISA/eLISA [32–35]. The wavelength of the observed waves scales with the length scale of the experiment. So, LISA satellite will detect longer waves than the ground based experiments. This means observing merging of super-massive black holes and earlier stages of smaller mergers. This hopefully allows tracking systems of e.g. merging black holes or neutron stars over several years. The idea is to first observe earlier, longer wavelength inspiral phase with LISA satellite and then catch the final phases down on Earth with LIGO/Virgo. This will allow for a direct test on theories of gravity and on our computational methods from weak through strong gravity.

European Space Agency's satellite mission Euclid [36] will tell more about dark energy and possible modified gravity. The null hypothesis, that the Universe is best described by Friedmann equations and ΛCDM, will be tested by two complementary measurements of the matter distribution in the Universe.

Firstly, Euclid will measure the typical amount of dark matter between us and galaxies at different redshifts. The amount is estimated by measuring distortion of pictures of galaxies; the higher the distortion the more there is lensing by dark matter along the way. Secondly, observing galaxies at different redshifts gives a measure of the development of the baryon acoustic oscillations. These oscillations appear as bubbles with walls of baryonic matter created in the early Universe. Pressure pulses from overdensities threw out baryonic matter at constant speed, leaving imprints in the CMB and the large scale structure. After decoupling from radiation baryonic matter became pressureless and since then the evolution of these bubbles has been due to gravity and dark energy only. Comparing these two pictures of expansion history will give a way to test our models of nature.

With the coming experiments and growing computational abilities we need better understanding of theories that we will want to test in the future. This thesis for its part will give a brief introduction to some of the models that have and continue to be under investigation by gravity research community. The work is sometimes cumbersome and the steps of progress are small but we still need to take them in order to improve our picture of the weakest of the interactions and eventually of the Universe.

Chapter 2

Slightly more technical introduction and basic concepts

In this chapter we dive into the mathematical formulation of gravitational theories. Mathematics is convenient for describing natural phenomena, for one, because there are regularities in natural phenomena that allow finding universal features or sometimes laws that reflect the behaviour of physical systems. These features or sometimes laws of nature are often most efficiently and most precisely described by mathematics.

Secondly, mathematics is also convenient by construction. An important motivation for the development of mathematics has been to describe physical world efficiently and precisely. Lengths, vectors, rates of change, and worldlines through spacetime all have their counterparts in mathematical formulation. These are also geometrical concepts that are useful for example in describing the movement of the planet Mercury around the Sun.

In this chapter we will discuss some basic concepts needed in gravity research. We will also go through some examples in order to underline and to clarify certain general features.

Throughout the whole thesis we will assume the Einstein summation convention, summing over repeated indices. We will also assume $(-,+,+,+)$ signature unless otherwise noted. We will also adopt units such that the speed of light in vacuum $c=1$ unless otherwise noted.

2.1 Basic concepts, the metric and the connection

There are several ways of formulating a theory of gravity. What is common for all of them is that they should agree with observations. A natural starting point for a geometric theory is a simple invariant observable, a spacetime interval or clock reading along the observer's worldline [37, 38]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ is *the metric tensor* of the spacetime, a symmetric rank two tensor field whose eigenvalues have the signs $(-,+,+,+)$. The intervals dx^μ are differential displacement vectors in the direction of an axis of the chosen system of coordinates. In this thesis we assume that such single tensor field can be found for any given observer.

Each observer defines a unique frame of reference, their rest frame, where they can measure ds^2 by a suitable idealised clock. Every other observer agrees on the value of the interval but not necessarily on its individual components. The coordinate time or spatial displacements during that interval (i.e. velocity) will be different for different observers in their own rest frames.

Einstein generalised Newton's first law to general frames of reference by stating that a freely moving particle follows a path that corresponds to a straight line in spacetime, *the geodesic*. That is, for a displacement between two spacetime events, A and B, the geodesic gives an extremum of the displacement

$$s_{AB} = \int_A^B ds. \tag{2.2}$$

The extremum is given by the geodesic equation [37]

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \tag{2.3}$$

where the spacetime connection $\Gamma^\mu_{\alpha\beta}$ has been introduced. To be more precise: In the geodesic equation s is an affine parameter of the worldline $x^\mu(s)$. An observer moving on a timelike geodesic can identify s with their clock reading.

In GR the connection is generated as the Christoffel symbol of the metric as

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\delta} \left(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right). \tag{2.4}$$

This is a choice that is not made in theories where the so-called Palatini variation is assumed. Rather, in those theories the connection is a priori assumed independent of the metric. We will focus on theories with independent connection in chapter 3.

In chapter 5 we present a model in which we introduce a second "metric" tensor to generate an independent connection. These models have been considered in the literature in order to generalise the Palatini approach. However, we show that introducing the new gravitational degrees of freedom leads to instabilities in the Hamiltonian. Namely, we find that the theory has a field decaying to lower energy states ad infinitum, emitting an infinite amount of energy. Therefore, these models can not be considered viable models of gravity.

Spacetime connection is the quantity that dictates the parallel transport of vectors. Mathematically it is associated with the covariant derivative (∇_μ), such that for a tensor $T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}$

$$\nabla_\mu T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m} = \partial_\mu T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m} + \sum_{i=1}^{n} \Gamma^{\alpha_i}_{\mu\nu} T^{\alpha_1\cdots\nu\cdots\alpha_n}_{\beta_1\cdots\beta_m} - \sum_{i=1}^{m} \Gamma^{\nu}_{\mu\beta_i} T^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\nu\cdots\beta_m}, \tag{2.5}$$

where the summation index ν represents summation over the contravariant and covariant components in place of α_i and β_i respectively. For example,

$$\nabla_\mu T^{\alpha}_{\beta_1\beta_2} = \partial_\mu T^{\alpha}_{\beta_1\beta_2} + \Gamma^{\alpha}_{\mu\nu} T^{\nu}_{\beta_1\beta_2} - \Gamma^{\nu}_{\mu\beta_1} T^{\alpha}_{\nu\beta_2} - \Gamma^{\nu}_{\mu\beta_2} T^{\alpha}_{\beta_1\nu}. \tag{2.6}$$

In a general (four-dimensional) gravity theory the field equations can be calculated from the action

$$\mathscr{S} = \int \mathrm{d}^4x \sqrt{-g} \mathscr{L}_G + \mathscr{S}_m(\Psi), \tag{2.7}$$

where $\mathscr{L}_G$ is the Lagrangian density of gravitational tensor, vector and scalar fields. The action for all the matter fields (Ψ) is given by $\mathscr{S}_m(\Psi)$. This matter Lagrangian may or may not depend on the gravitational fields, often depending on the choice of frames.

The equations of motion are then obtained as Euler-Lagrange equations for the given Lagrangian density and the degrees of freedom. In other words, the gravitational equations are obtained by varying the action with respect to the gravitational field or fields and finding the extrema of the action as

$$\delta\mathscr{S} = 0. \tag{2.8}$$

What distinguishes between different theories is the mathematical structure of the spacetime. It is determined by the gravitational fields and their properties such as coupling to matter. For general relativity the only field is the rank two metric tensor $g_{\mu\nu}$. The metric tensor gives the properties of the spacetime, the lengths, and the geodesics. Other theories may have additional scalar, vector, or tensor fields that affect these properties.

2.1.1 Field equations

As an example of gravitational theory and the action principle, let us consider a (Jordan-)Brans-Dicke [39] type Scalar-Tensor theory with the action

$$\mathscr{S} = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} \left(\phi R - \omega \frac{\phi_{,\mu}\phi^{,\mu}}{\phi} - V(\phi) + 16\pi G \mathscr{L}_m(\Psi, g_{\mu\nu}) \right). \tag{2.9}$$

Here ϕ is a scalar function, for example of the Ricci scalar as in Palatini f(R) theories in chapter 3. The *Brans-Dicke coupling constant* ω is a free parameter that distinguishes between different theories, and $\mathscr{L}_m$ contains all non-gravitational (matter) fields. In the original paper [39] Brans and Dicke consider a theory without the potential $V(\phi)$ but for many applications it is important.

In many cases the constants are chosen so that formulas look as simple as possible. In (2.9) we have chosen to keep the gravitational constant G. The reason is to highlight the interpretation that in scalar-tensor theories the gravitational coupling strength is not a constant but a function of the scalar field instead. This can be seen for example from the field equation (2.10) below. Sometimes in the literature the notation $G(\phi)$ is used. In the present thesis G will denote a constant of nature whereas $G(\phi)$ is a function of the scalar. For example in (2.10) and in the original paper [39] the coupling strength $G(\phi) \propto \phi^{-1}$.

The variational degrees of freedom are the components of the metric tensor and the scalar field ϕ. Variation with respect to both the metric and the scalar and their derivatives gives the field equations respectively [40]

$$\begin{aligned} &G_{\mu\nu} = \frac{8\pi G}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right) + \frac{1}{\phi}\left(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi\right) - \frac{V}{2\phi} g_{\mu\nu}, \\ &2\omega\frac{\Box\phi}{\phi} - \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2} + R - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \end{aligned} \tag{2.10}$$

This follows the original formulation of [39]. Sometimes it is convenient to use the trace of the first equation to eliminate the Ricci scalar from the second. In that case the equation for the field ϕ reads

$$\Box\phi = \frac{1}{2\omega + 3}\left(8\pi G T + \frac{\mathrm{d}V}{\mathrm{d}\phi}\phi - 2V\right), \tag{2.11}$$

where T is the trace of the energy-momentum tensor. A word of caution though: Palatini f(R) theories are equivalent to Brans-Dicke theories with $\omega = -3/2$. Therefore some of the expressions, such as (2.11), have to be used with caution. For example, in this case the equivalent expression for (2.11) reads

$$8\pi G T + \frac{\mathrm{d}V}{\mathrm{d}\phi}\phi - 2V = 0. \tag{2.12}$$

While exceptions exist, most often these equations are second order differential equations for the metric in (2.1). Therefore their solution, $g_{\mu\nu}$, is directly related to the observable clock reading ds^2.

2.1.2 Conformal couplings

There is some freedom on the choice of frames in which the theory is written.[1] These frames correspond to different choices of the metric and the connection. The freedom is mirrored in the form of the geodesics and the gravitational degrees of freedom coupling to matter and light. This easily causes confusion about the physical interpretation of the theory that one would intuitively expect to be invariant with respect to the choice of variables. Actually, there has been some confusion regarding which geodesics matter and light follow (see e.g. [1, 41] and references therein).

In the action (2.9) the gravitational scalar field couples to the geometry through the Ricci scalar R. In this form, or frame, matter follows the geodesics which are affected by the scalar field. However, it possible to redefine the metric via a Weyl rescaling (2.13) such that, in the resulting Lagrangian, a redefined scalar field couples to matter fields instead of the curvature R. In this frame a fifth force may appear which distorts matter from geodesics.

The frame in which matter fields follow metric geodesics is called the *Jordan frame* and the one in which ϕ decouples from R the *Einstein frame*. In other words, in the Jordan frame the gravitational scalar field couples minimally to matter. This is the case for the Lagrangian of our earlier example (2.9). Moreover, in this frame the gravitational coupling strength $G(\phi) \propto \phi^{-1}$. In the Einstein frame, the coupling strength $G(\phi)$ is constant, and the scalar field couples minimally to the curvature, but to matter fields instead.

Bekenstein [42] puts it: "Usually one of these describes gravitation while the other defines the geometry in which matter plays out its dynamics. The strong equivalence principle is violated by all these two-geometries theories, but they usually preserve weak equivalence." We will discuss the weak and the strong equivalence principles in more detail in section 2.3

For example, in Brans-Dicke gravity this change of frames means writing the Lagrangian in terms of a rescaled metric $\hat{g}_{\mu\nu}$, given by a conformal transformation

$$\hat{g}_{\mu\nu} \equiv \Omega^2(\phi) g_{\mu\nu}, \tag{2.13}$$

where in case of Brans-Dicke theory $\Omega^2 = G\phi$. With this rescaling the Brans-Dicke action (2.9) can be written as

$$\mathscr{S} = \int \mathrm{d}^4x \sqrt{-\hat{g}} \left[\frac{\hat{R}}{16\pi G} - \frac{1}{2}\varphi_{,\mu}\varphi^{,\mu} - \hat{V}(\varphi) + \exp\left(-8\sqrt{\frac{\pi G}{2\omega + 3}}\,\varphi \right) \mathscr{L}_m(\Psi, \hat{g}_{\mu\nu}) \right], \tag{2.14}$$

[1] Note on terminology: Previously we referred to different systems of coordinates as *frames* of reference. In the present context the different *frames* correspond to the choice of the metric and the connection. This choice can affect for instance the interpretation interactions.

where the new field variable is defined as

$$\varphi \equiv \sqrt{\frac{3+2\omega}{16\pi G}} \ln(\phi) \tag{2.15}$$

and its potential

$$\hat{V}(\varphi) = \frac{V(\phi)}{\phi^2}. \tag{2.16}$$

In action (2.14) the new gravitational scalar field couples to matter instead of geometry via the Ricci scalar. The conformal scaling (2.13) gives the transformation from one frame to another as a redefinition of variables. Thus, one might expect that physics does not depend on the choice of frames. After all, physics should not depend on the variables with which it is written. However, there has been some debate on whether the two frames are indeed physically equivalent [43–45]. The debate considers the equivalence of quantum corrections and we will not go into details here. The consensus is that classically or to "tree level" different conformal frames are equivalent.

From a spacetime interval $\mathrm{d}s$, the conformal transformation (2.13) can be interpreted as rescaling of units [39, 46]. Namely, the observable

$$ds^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \hat{g}_{\mu\nu}\,\mathrm{d}\hat{x}^\mu\,\mathrm{d}\hat{x}^\nu, \tag{2.17}$$

only if $\mathrm{d}\hat{x}^\mu = \mathrm{d}x^\mu/\Omega(\phi)$. Thus, for example lengths depend on the position via $\phi(x)$. In [47] the authors argue that instead of ds^2 the proper invariant quantities should be dimensionless, such as $M_p^2\,\mathrm{d}s^2$. Moreover, the whole Lagrangian should be written using dimensionless quantities so that it would be invariant under conformal transformations.

In chapter 4 and in paper [3] we study a model where a more general coupling between the two frames is used. In addition to the conformal coupling, in this so called *disformal* coupling the transformation also depends on the field derivatives. Hence, matter also couples to the derivatives of the field which leads to different predictions in various physical situations.

2.2 Compact objects

2.2.1 Schwarzschild and Kerr black holes

General relativity can be thought of as a special case for a scalar-tensor theory. In it the scalar field is constant $\phi \equiv 1$, its derivatives vanish, the potential is the cosmological constant ($V = 2\Lambda$). The field equations become *the Einstein equations*

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}. \tag{2.18}$$

As stated before, the solution to the field equation is the metric tensor, which corresponds to the matter distribution described by $T_{\mu\nu}$. Some of the most famous and also the most useful for the considerations in this thesis are the Schwarzschild and the Kerr solutions of GR.

The Schwarzschild metric [48] was the first nontrivial solution to be discovered. It describes the spacetime outside a static and spherically symmetric massive object of mass M. The Schwarzschild metric provides a sufficient first approximation to describe many physical situations, such as the solar system. Using the Schwarzschild metric the line element reads

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\,\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\,\mathrm{d}\varphi^2, \tag{2.19}$$

where $r_s = 2GM$ is the Schwarzschild radius.

It took almost half a century for an exact solution to be found for a rotating axisymmetric body. The solution is the Kerr metric [49]

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s r}{\rho^2}\right)\mathrm{d}t^2 + \frac{\rho^2}{\Delta^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\vartheta^2 + \left(r^2 + \alpha^2 + \frac{r_s r\alpha^2}{\rho^2}\sin^2\vartheta\right)\sin^2\vartheta\mathrm{d}\varphi^2 - \frac{2r_s r\alpha\sin^2\vartheta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi, \tag{2.20}$$

where $\alpha = J/M$ is the angular momentum divided by mass of the object, $\rho^2 = r^2 + \alpha^2\cos^2\vartheta$ and $\Delta = r^2 - r_s r + \alpha^2$. In the limit where the angular momentum vanishes, i.e. the black hole becomes non-rotating, the above of course reduces to (2.19).

These two metrics are often used as background metrics, even when considering modified gravity rather than GR. This requires the assumption that e.g. the backreaction of matter around the black hole is small. In other words, the effects of surrounding matter or the scalar field are only perturbative corrections to the background metric.

2.2.2 PPN parametrisation

Because of the sheer amount of different kinds of extended theories of gravity there has been attempts to create unified frameworks in which to compare the properties of different theories [50–54]. A way to proceed in this is to provide a model-independent parametrisation for a given physical system, for example gravitational field around a compact object. Then, on one hand, it is possible to derive the parameter values for a given theory. On the other hand it becomes possible to compare these parameter values to measurements and observations. Thus, restrictions can be established that each theory must satisfy in order to be considered viable.

Systems of compact objects are one of the few laboratories to test gravitational effects observationally. Our own Solar system is an example of such a system and originally the only astrophysical environment in which to compare gravity theory with measurements [16, 18, 55, 56].

Table 2.1: Post-Newtonian parameters, their limits and their significance [18]. For GR $\gamma = \beta = 1$ and the rest are zero.

Parameter	Limit	Effect	What it measures
$\gamma - 1$	2.3×10^{-5}	Time delay	Space-curvature produced by unit rest mass
$\beta - 1$	2.3×10^{-4}	Nordvedt effect	"Nonlinearity" in the superposition law for gravity
ξ	10^{-3}	Earth tides	Preferred-location effects
α_1	10^{-4}	Orbital polarisation	Preferred-frame effects
α_2	4×10^{-7}	Spin precession	
α_3	4×10^{-20}	Pulsar acceleration	
α_3			Violation of conservation of total momentum
ζ_1	2×10^{-2}		
ζ_2	4×10^{-5}	Binary acceleration	
ζ_3	10^{-8}	Newton's 3rd law	
ζ_4	–		

The Parametrized Post-Newtonian formalism [50] was constructed for comparing observations with theoretical predictions, to test the equivalence principle, and later to constrain deviations from it. In this formalism the spacetime metric is expanded as a series around a known background, the Newtonian gravity. The metric can be written in terms of ten parameters, the post-Newtonian or PPN parameters. The formalism is known as the Parametrized post-Newtonian (PPN) formalism. These parameters determine the corrections to the Newtonian metric.

PPN approximation assumes a weak field and slow velocities. That is, as measured in terms of small book-keeping parameter ϵ it is assumed that $U \sim \Pi \sim v^2 \sim p/\rho \sim m/r \sim \epsilon$ and likewise $v^i \sim |d/dt|/|d/dx| \sim \epsilon^{1/2}$. Here U is the Newtonian potential, Π is the internal energy per unit rest mass (pressure, temperature etc. non-rest-mass-type energy). The metric tensor, equations of motion and other objects under consideration are then written as a series expansion in powers of ϵ.

The ten parameters and their current observational values [18] are listed in table 2.1. Using the

PPN parameters the metric, in "standard PPN gauge" is

$$\begin{aligned}
g_{00} &= -1+2U-2\beta U^2-2\xi\Phi_W+(2\gamma+2+\alpha_3+\zeta_1-2\xi)\,\Phi_1+2\,(3\gamma-2\beta+1+\zeta_2+\xi)\,\Phi_2+ \\
&+2\,(1+\zeta_3)\,\Phi_3+2\,(3\gamma+3\zeta_4-2\xi)\,\Phi_4-(\zeta_1-2\xi)\,\mathcal{A}-(\alpha_1-\alpha_2-\alpha_3)\,w^2U- \\
&-\alpha_2 w^i w^j U_{ij}+(2\alpha_3-\alpha_1)\,w^iV_i+\mathcal{O}(\epsilon^3), \\
g_{0i} &= -\frac{1}{2}\,(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi)\,V_i-\frac{1}{2}\,(1+\alpha_2-\zeta_1+2\xi)\,W_i-\frac{1}{2}\,(\alpha_1-2\alpha_2)\,w^iU- \\
&-\alpha_2 w^j U_{ij}+\mathcal{O}(\epsilon^{5/2}), \\
g_{ij} &= (1+2\gamma U)\,\delta_{ij}+\mathcal{O}(\epsilon^2).
\end{aligned} \tag{2.21}$$

Here the Newtonian potential

$$U=\int\frac{\rho'}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x' \tag{2.22}$$

and the post-Newtonian potentials are listed in appendix A.1. In chapter 3 the post-Newtonian metric and equations of motion, the geodesic equation (2.3), are calculated for the Palatini $f(R)$ theory.

For GR the parameters $\gamma=\beta=1$ while the rest are zero. These parameters are traditionally constrained by solar system observations. However, increasingly accurate measurements from astrophysical systems require using beyond the first-order post-Newtonian approximation (see for example [24, 25]). Such calculations are an active field of study both for GR [57–60] and for alternative theories [1, 6, 61, 62].

2.3 Equivalence principles and the difference between modified gravity and dark energy

The observed accelerating expansion of the universe [63, 64] has launched an expansion also in the number of research articles concerning gravity theories. In the current standard ΛCDM [2] cosmology the acceleration is driven by dark energy or a cosmological constant, Λ, in Einstein's equations. The other possibility for explaining this acceleration is to modify Einstein gravity, for example into a scalar-tensor gravity. General relativity equipped with a cosmological constant and scalar-tensor gravity are examples of theories in two distinct categories for explaining the acceleration. Cosmological constant represent a model of dark energy while scalar-tensor theories are examples of modified gravity.

A classification for distinguishing between dark energy and modified gravity models is suggested in [10]. The division is based on whether or not the theory preserves the strong equivalence principle.

[2] The letter Λ stands for the cosmological constant and CDM for cold dark matter. These two components constitute the energy content of the Universe.

Theories for which the strong equivalence principle holds to measurable accuracy, the expansion is driven by dark energy. In the other case the theories fall in the class of modified gravity.

We will next go through the different formulations of equivalence principles and then come back to the two classes of theories.

We will discuss three different formulations of equivalence principles. The common denominator is the idea that with respect to free fall, all physical observers are equivalent. Three different versions are typically presented, *the weak*, *the strong* and *the Einstein equivalence principles* [18, 53]

Weak equivalence principle. The first of these principles, the weak equivalence is the one familiar from Newtonian physics. It simply states that the inertial mass and the weight of any test particle are equivalent [65]. This statement is equivalent of saying that all test bodies will fall on same trajectories and undergo the same acceleration regardless of their composition or structure [53]. That is, the trajectory of a body is dictated by its initial position and velocity, not for instance its mass.

Let us clarify the equivalence of the two statements above. The universality of free fall was not invented by Newton. Even before Galilei there had been experiments of dropping objects from high places. Galilei conducted his experiments on an inclined plane and concluded that the rate of fall was independent of the mass of the body. Newton formulated mathematically the laws of motion. According to them the acceleration due to the gravitational field is

$$\ddot{\mathbf{x}} = -G\frac{M}{r^2}\hat{r}. \tag{2.23}$$

Here $\ddot{\mathbf{x}}$ is the acceleration (vector) of the object in the gravitational field, M is the mass of the source, r is the relative distance between the the source and the object ($\hat{r}$ is the unit vector directed from the source to the object), and G is a constant of proportionality. However, this result requires the assumption that gravitational weight and the inertial mass of the object are equivalent. The gravitational force is proportional to the weight whereas the acceleration in Newton's second law is proportional to inertial mass. If the two are proportional to one another in a similar manner for all observers, the ratio of the two can be absorbed into the gravitational constant. Equation (2.23) follows and with this assumption the mass of the observer does not contribute to its acceleration.

Einstein equivalence principle. Einstein wanted all observers to be equivalent with respect to one another. Namely, he wanted to broaden the weak equivalence principle to non-inertial reference frames. What is called the principle of relativity [66] states that gravitational field is equivalent to an accelerated reference frame. This requires that the mass and weight are directly proportional. This principle is generally known as the Einstein equivalence principle and summed up in [53] it states that

1. [Weak equivalence principle] is valid.
2. The outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed.
3. The outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed.

Strong equivalence principle. The next generalisation is to include also self-gravitating objects into the equivalent frames and gravitational experiments. This includes physical bodies beyond test-bodies such as planets, stars or neutron stars, for which inner gravitational effects are non-negligible. This also encompasses gravitational experiments such as the Cavendish experiment, which first measured the gravitational force between masses other than the Earth. Will summarises the strong equivalence principle [18] as

1. [Weak equivalence principle] is valid for self-gravitating bodies as well as for test bodies.
2. The outcome of any local test experiment is independent of the velocity of the (freely falling) apparatus.
3. The outcome of any local test experiment is independent of where and when in the universe it is performed.

The difference between this and the Einstein equivalence principle is the inclusion of self-gravitating objects and gravitational experiments.

It is clear that if for example the coupling strength varies significantly from place to place, the strong equivalence principle does not hold. This is the case for many of the scalar-tensor theories, including the ones studied in chapter 4 in this thesis.

As mentioned earlier, the strong equivalence principle can be used to classify theories as models with dark energy driving the accelerated expansion, and as theories of modified gravity [10]. As an example of dark energy the authors name quintessence in which an extra scalar field is added to GR instead of the cosmological constant [67–69]. The scalar field is much like the cosmological constant of dark energy, only its value varies with time. In [10] they relax the requirement of exact temporal invariance and concentrate on the free fall. In this case the effect of the scalar dark energy is negligible on small scales and does not contribute to the "local test experiments" .

An example of modified gravity is the Brans-Dicke gravity in section 2.1.1. In the action (2.9), the additional scalar field also couples to the geometry (the Ricci scalar). In Brans-Dicke theories objects such as neutron stars may carry scalar charge while black holes do not. This can lead to conflict with the first condition of the strong equivalence principle. Therefore, Brans-Dicke gravity falls under the class of modified gravity.

2.4 Outline

For the rest of this thesis we will concentrate on three different extensions of GR: Palatini f(R), Scalar-tensor, and a certain class of bimetric theories.

- In chapter 3 and in [1] we consider Palatini f(R) theories. It is a class of theories in which the spacetime metric and the connection are a priori independent degrees of freedom. The Lagrangian of the theories we consider is of the form

  $$\mathscr{L}_G = f(\hat{R}) = \hat{R} + \phi(\hat{R}), \tag{2.24}$$

  where $\hat{R}[\hat{\Gamma}]$ is the Ricci scalar of the independent metric and $\phi(\hat{R}[\hat{\Gamma}])$ is a differentiable function of $\hat{R}[\hat{\Gamma}]$. For GR $\phi(\hat{R}) \equiv 0$ and the Einstein equations are obtained regardless of whether the connection is independent or not.

  The equations of motion for astrophysical objects had not been studied very far for this class of modified gravity. In [1] we show that up to second order in the post-Newtonian expansion, the equations remain intact except for a scaling of the gravitational mass of the objects. This result is in line with the literature [6]. Their results use the scalar-tensor notation, however, and therefore are be to used with caution (see remark after (2.11)).

- In chapter 5 and in paper [2] we consider a class of theories where the Palatini principle is taken one step further. In these theories the independent connection is thought to be constructed by an additional rank two tensor field. We call them theories that follow the bimetric variational principle to separate them from a wider class of bimetric theories.

  In these theories the Lagrangian is constructed of both the gravitational metric as well as the new independent metric. Namely,

  $$\mathscr{L}_G = \hat{R} + R, \tag{2.25}$$

  where $R[g_{\mu\nu}]$ is the Ricci scalar of the physical metric and $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}[\hat{g}_{\mu\nu}]$ is the Ricci scalar of the independent connection contracted with the physical metric. The equations of motion are then obtained by varying the action with respect to both tensors instead of the metric and the connection, as in Palatini theories.

  These theories have been thought to be interesting as a possible candidate for a class of bimetric gravity. To the first order perturbations around flat spacetime they had been shown to avoid problems that often bother bimetric theories. However, in [2] we show that in such theories propagating ghost degrees of freedom are unavoidable. Thus, such theories cannot be considered viable theories of gravity.

- In chapter 4 and in [3] we consider scalar-tensor gravity. In these theories there is a gravitational scalar field in addition to the metric tensor. We study a class of models where the scalar field couples to matter via the so called *disformal coupling* [42, 70]. This coupling is obtained by a transformation between the Einstein frame and the Jordan frame. The most general transformation between the two frames is a redefinition of the metric, the disformal transformation, such that

$$\hat{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X)\phi_{,\mu}\phi_{,\nu}. \tag{2.26}$$

  Here $g_{\mu\nu}$ is the Jordan frame metric the geodesics of which matter follows and $\hat{g}_{\mu\nu}$ is the Einstein frame metric. The two functions $C(\phi, X)$ and $D(\phi, X)$ are the model specific conformal and disformal coupling functions. The kinetic term of the scalar field $X = -(\partial\phi)^2/2$.

  The Lagrangian for the Jordan frame, that is, using the physical metric, is

$$\mathscr{L}_J = \frac{f(\phi)R}{16\pi G} + q^{\phi}(\phi, X) + \mathscr{L}_m(\Psi, g_{\mu\nu}), \tag{2.27}$$

  while for the Einstein frame, using the redefined metric, the Lagrangian couples to matter fields via $\hat{g}_{\mu\nu}$ as

$$\mathscr{L}_E = \frac{\hat{R}}{16\pi G} + p^{\phi}(\phi, X) + \mathscr{L}_m(\Psi, \hat{g}_{\mu\nu}). \tag{2.28}$$

  These theories have gained interest during the past ten years (see for example [71, 72] for review of different applications of disformal couplings). In chapter 4.2 we also present a list of various ideas of where to apply these theories. In our article [3] we consider black holes with surrounding matter, such as an accretion disk. We study such systems for both Schwarzschild and Kerr black holes. The idea of the investigation was to find possible constraining effects for the disformal coupling. We conclude that depending on the coupling strength the disformal coupling can either stabilise or destabilise such a system.

This is by no means an exhaustive list of approaches to modifying general relativity. On the contrary, a vast amount of modified gravity theories exist. These include such as discussed in [73, 74] where a vector field is added, or the class of massive gravity models briefly discussed in chapter 5 to name a few. All of these different models have their motivation, most focus on fixing some particular problem of Newtonian gravity, or of GR. The wide variety of different modified gravity theories is an implication of the room that is left for speculation on the nature of the gravitational interaction. To date, we are not sure of the number of gravitational fields, and much less so of their mathematical formulation. In this thesis we will focus on the three approaches listed above; other models exist but remain out of the scope, and we will not go further into any details here.

Chapter 3

Palatini f(R) gravity

One way to to generalise general relativity is to relax some of the assumptions about spacetime geometry. In differential geometry there are two properties that define the structure of a manifold. First, there is the metric tensor, which defines distances between events. The other is the connection, which gives parallel transport along worldlines. A priori there is no reason for these two to depend on one another, this is just an assumption made in GR.

Another assumption to modify concerns the form of the action. Theories that generalise GR by relaxing either one or both of these assumptions form a particularly interesting class of theories. For example, within these classes there are models that have exponentially expanding or inflating universe without dark energy. More often than not, there are of course other shortcomings to these models that rule them out as realistic alternatives to GR. Studying them may nonetheless give insight into what gravitational phenomena beyond GR and suggest where one may look for the possible deviations.

In this chapter we will concentrate on extensions of general relativity in which the spacetime connection is a priori independent of the spacetime metric while the action is a functional of the Ricci scalar. In these theories the equations of motion are obtained by varying the action with respect to both the metric and the connection.

3.1 Palatini gravity

Palatini variational principle was considered already in 1925 by Einstein [75] (for a history of "Palatini's method" see [76]). Already then it was noted that in the case of Einstein-Hilbert action one ends up with the Einstein equations regardless of whether one assumes the connection to be of the Levi-Civita form (Christoffel symbol) and a functional of the metric or takes it as an independent degree of freedom. However, for more general actions the emerging equations of motion can be non-equivalent, as will be shown later in this chapter.

In [1] we study one particular class of theories with a generalised action. In this class the Ricci scalar of Einstein-Hilbert action is replaced by some sufficiently differentiable function of the scalar. The function is usually denoted by $f(R)$ and the theories hence dubbed $f(R)$-*theories of gravity*.

Some of the most common choices for $f(R)$ are quadratic $R+\alpha R^2$ and the inversely proportional $R+\beta/R$. These are of course also the algebraically simplest choices and the first terms in a series expansion in some suitable small quantity.

There are three different common variations of $f(R)$ theories depending on the role of the connection. The first is to assume that the connection is the Christoffel symbol of the metric. Such theories are called *the metric* $f(R)$ theories. The second is to assume the connection to be a priori independent of the metric. Such theories are called *the Palatini* $f(R)$ theories. These two are the theories we will concentrate on for the rest of this chapter. As we will show later in this chapter, in Palatini $f(R)$ theories the a priori independent connection turns out to be an auxiliary field. Therefore, the connection defining the parallel transport and the covariant derivative "remains the Levi-Civita connection of the metric" [77]. The third possibility is to drop the remaining assumptions about the relation between the connection and the metric. This approach is known as *the metric-affine formalism*. In this case also the covariant derivative and parallel transport are given by the independent metric, which results in the independent connection to appear also in the matter Lagrangian.

Historically the motivation for studying $f(R)$ theories was to merely generalise the Einstein-Hilbert action for academic purposes. Indeed, after the introduction of general relativity it was questionable whether or not any of its predictions would even be observable in the near future. Detecting small deviations from GR was an idea even further away.

Since the birth of the inflationary paradigm at the end of 1970's and in the beginning of 1980's [78–80] (see e.g. [81, 82] for reviews) $f(R)$ type theories have gathered interest also in more "practical" applications. In the so called Starobinsky model [80] the inflating expansion of the Universe is achieved by a Lagrangian where

$$f(R) = R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}. \tag{3.1}$$

Though not strictly a function of the Ricci scalar alone, this model is considered to be one of the first applications of $f(R)$ type action in an actual physical setup. Later on, as the current accelerated expansion of the universe has become widely accepted, the focus has shifted from the inflating geometries of early times to the late time expansion (see [77, 83] for reviews).

3.2 Field equations

The field equations of $f(R)$ type of gravity theories are given by the action

$$\mathscr{S} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}f(R) + \mathscr{S}_m, \tag{3.2}$$

where $\kappa = 8\pi G$, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar contracted from the corresponding Ricci tensor with the inverse metric $g^{\mu\nu}$, g is the determinant of the metric tensor and S_m is the action for the matter fields. $f(R)$ can in principle be any (mostly) analytic function of the curvature scalar R. The function $f(R) = R$ gives the Einstein-Hilbert action of general relativity.

The Ricci tensor is constructed of the connection as

$$R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu}. \tag{3.3}$$

For the Palatini theories the Ricci tensor is thus independent of the physical spacetime metric. From here on I will denote the independent connection and the functions of it by a hat (e.g. $\hat{\Gamma}^\alpha_{\mu\nu}$ and $\hat{R}$).

In the *Palatini* formulation of $f(R)$ theories the Ricci curvature tensor $\hat{R}_{\mu\nu} = \hat{R}_{\mu\nu}[\hat{\Gamma}^\alpha_{\mu\nu}]$, but the corresponding curvature scalar, $\hat{R} \equiv g^{\mu\nu}\hat{R}_{\mu\nu}$, is also a function of the metric. Indices are raised and lowered with the metric tensor and its inverse. The matter Lagrangian, S_m, doesn't depend on the independent connection but matter follows the geodesics of the unhatted metric connection $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu}[g_{\mu\nu}]$.

In the Palatini formalism variation with respect to the metric and the connection yield the field equations

$$f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = \kappa T_{\mu\nu} \tag{3.4}$$

$$\hat{\nabla}_\alpha\left(\sqrt{-g}f'(\hat{R})g^{\mu\nu}\right) = 0 \tag{3.5}$$

respectively. The notion $f'(\hat{R})$ refers to differentiation with respect to the argument. It is also useful to write down the trace of (3.4) as

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = \kappa T. \tag{3.6}$$

This trace equation is now an algebraic equation of R and can be used to solve the curvature scalar in terms of T, formally $\hat{R} = \hat{R}(T)$.

Next we want to write the field equation (3.4) using the metric Einstein tensor $G_{\mu\nu}(g_{\mu\nu})$. For this we define a metric $\hat{g}_{\mu\nu}$ which is related to $g_{\mu\nu}$ via conformal transformation, namely

$$\hat{g}_{\mu\nu} \equiv f'(\hat{R}) g_{\mu\nu}. \tag{3.7}$$

Now it can be shown that

$$\sqrt{-\hat{g}}\,\hat{g}^{\mu\nu} = \sqrt{-g}\,f'(\hat{R}) g^{\mu\nu} \tag{3.8}$$

and so the field equation (3.5) gives the usual definition of a Christoffel symbol (2.4) for the independent connection (hence the hat in $\hat{g}_{\mu\nu}$)

$$\hat{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\hat{g}^{\alpha\lambda}\left(\hat{g}_{\mu\lambda,\nu} + \hat{g}_{\lambda\nu,\mu} - \hat{g}_{\mu\nu,\lambda}\right) \tag{3.9}$$

The conformal relation (3.7) then gives the relation between the independent and the metric connection and so the metric Ricci tensor $R_{\mu\nu}$ and the Ricci tensor constructed of the independent connection[1]

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2}\frac{1}{(f'(\hat{R}))^2}(\nabla_\mu f'(\hat{R}))(\nabla_\nu f'(\hat{R})) - \frac{1}{f'(\hat{R})}\left(\nabla_\mu\nabla_\nu + \frac{1}{2}g_{\mu\nu}\Box\right) f'(\hat{R}), \tag{3.10}$$

where the covariant derivatives are calculated using the metric connection. The curvature scalar is now

$$\hat{R} = R + \frac{3}{2}\frac{1}{(f'(\hat{R}))^2}(\nabla_\mu f'(\hat{R}))(\nabla^\mu f'(\hat{R})) - \frac{3}{f'(\hat{R})}\Box f'(\hat{R}). \tag{3.11}$$

Thus we have algebraic relations between the metric curvature variables and the variables constructed of the independent connection. The relations can then in principle be used to write the field equations (3.4) in terms of the metric curvature, which is convenient when we consider the equations of motion of compact objects. Also, the field equations can now be brought to Einstein-like form

$$\begin{aligned} G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad &= \quad \frac{\kappa}{f'(\hat{R})}T_{\mu\nu} - \frac{g_{\mu\nu}}{2}\left(\hat{R} - \frac{f(\hat{R})}{f'(\hat{R})}\right) + \frac{1}{f'(\hat{R})}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right) f'(\hat{R}) - \\ &- \frac{3}{2}\frac{1}{f'(\hat{R})^2}\left[\left(\nabla_\mu f'(\hat{R})\right)\left(\nabla_\nu f'(\hat{R})\right) - \frac{1}{2}g_{\mu\nu}\left(\nabla f'(\hat{R})\right)^2\right] \end{aligned} \tag{3.12}$$

As will later be shown the new degree of freedom does not introduce new dynamics to the theory and hence the modification of the right hand side of (3.12) can really be interpreted as modification of the source and scaling of the gravitational constant.

[1] This equation differs somewhat from [77], probably due to a typo in the reference. However, we agree on the curvature scalar in the following Eq. (3.11).

3.3 Equivalence between scalar-tensor and $f(R)$ theories

Now we come to the subtlety that was mentioned earlier in section 2.1.1, which requires some caution when performing the actual calculations. Namely, there is an equivalence between $f(R)$ theories and scalar-tensor gravity. This can be seen for example by comparing the field equations of Brans-Dicke theories (2.10)

$$\begin{aligned}&G_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right) + \frac{1}{\phi}\left(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi\right) - \frac{V}{2\phi}g_{\mu\nu},\\&2\omega\frac{\Box\phi}{\phi} - \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2} + R - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0.\end{aligned} \tag{3.13}$$

and $f(R)$ theories [84]. For Palatini $f(R)$ theories comparing Brans-Dicke equations one can read from equations (3.12) that

$$\begin{aligned}\phi &= f'(\hat{R}),\\ \omega &= -\frac{3}{2},\\ V &= f'(\hat{R})\hat{R} - f(\hat{R}).\end{aligned} \tag{3.14}$$

In other words, Palatini $f(R)$ gravity is equivalent to Brans-Dicke gravity with a potential and parameter $\omega = -3/2$. For this reason, some caution is required when studying these theories using the Brans-Dicke formulation. For example, often in the literature the post-Newtonian equations contain terms multiplied by $(3 + 2\omega)^{-1}$ (e.g. [6]). This multiplier obviously diverges at the limit $\omega \to -3/2$.

With a similar kind of procedure it can be shown that the metric $f(R)$ theories are also equivalent to Brans-Dicke theories. Compared to Palatini theories the difference is that for the metric $f(R)$ theories the Brans-Dicke parameter $\omega = 0$.

Going back it is then possible to write the Brans-Dicke action

$$\mathscr{S} = \frac{1}{2\kappa}\int \mathrm{d}^4x\left(\phi R - \frac{\omega}{\phi}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right) + \mathscr{S}_m. \tag{3.15}$$

The equivalence is now easily checked by substituting ϕ, ω and $V(\phi)$ from (3.14) and the Ricci scalar from (3.11) for Palatini theories.

Hastily and only based on the Lagrangians, one might be tempted to declare the field ϕ non-dynamical in the metric $f(R)$ theories, where the derivative terms vanish due to $\omega = 0$. Likewise, one might expect the field to be dynamical in the Palatini formulation where $\omega = -3/2$. However, the opposite is true. To this end, consider the equation (2.11) for the scalar field. For Palatini theories this equation reduces to

$$8\pi GT + \frac{\mathrm{d}V}{\mathrm{d}\phi}\phi - 2V = 0 \tag{3.16}$$

and for metric theories

$$\Box\phi = \frac{1}{3}\left(8\pi GT + \frac{\mathrm{d}V}{\mathrm{d}\phi}\phi - 2V\right) \tag{3.17}$$

corresponding to the values of the Brans-Dicke parameter. These equations then give the dynamics for the scalar field $\phi = f'(R)$.

From (3.16) it is clear that for the Palatini $f(R)$ theories the field is non-dynamical. On the other hand, this could have been guessed already from the fact that the "independent" Ricci scalar $\hat{R}(T)$ is an algebraic function of the trace of the energy-momentum tensor (3.6). This enables the removal of the independent connection from the field equation (3.12) altogether. Hence, the independent connection is just an auxiliary field and not an actual new dynamical degree of freedom.

Whether to study theories in the $f(R)$ or in the scalar-tensor formulation is more or less a matter of taste and convenience. There may be cases where it is easier to use the formulation with an extra scalar field and others where using a function of the Ricci scalar makes things simpler. However, a healthy amount of caution is in order when adopting results from the literature. As was previously seen with the scalar field kinetics, not everything is as it appears on the first sight. Moreover, many reported formulae of scalar-tensor theories do not hold as such when the Brans-Dicke parameter $\omega = -3/2$, as happens to be the case for Palatini $f(R)$ theories. In fact, it can be argued that many Palatini $f(R)$ models can not be taken as the $\omega \to -3/2$ limit of Brans-Dicke theories [85]. At least the limit should be applied where the divergence does not pose a problem.

In article [1], we derive for the Palatini $f(R)$ theories the first orders of the post-Newtonian expansion of the metric, and the equations of motion. The metric, the spacetime connection together with the geodesic equations (equations of motion for compact objects) are expanded in powers of velocity, the ratio of the sizes of the objects and their relative distances, the ratio of pressure and density, and other such small deviations from flat and empty space. Basically, these relate to the characteristic properties of objects inside the Solar system. The main result is that the modifications to Einstein gravity show as a rescaling of masses of the objects, or equivalently of the coupling strength as $G(\phi) \propto 1/f'(R)$.

Chapter 4

Disformal couplings in scalar-tensor gravity

There are two ways to develop theoretical physics. One way is to "make things as simple as it can be but not simpler"[1]. That is, to write physics as simply as possible and only add complicating components when absolutely necessary in order not to be in conflict with observations. The other way is to formulate the most general theories possible and then try to constrain them by observations.

An example of the first way is general relativity and the cosmological constant. It was well known that a constant term could be added to the Einstein equations but it was not until the accelerated expansion of the universe was observed that we found actual use for it. It had to be added to the simplest theory to explain the expansion.

Another way is to start with the very general and then constrain the different parts by measurements. In this section we will follow this approach and introduce the most general scalar-tensor theory with second order field equations. We will then move on to discuss the most general transformation between the Einstein frame and the Jordan frame.

These general theories and properties are interesting in that they encompass a large number of theories, that can thus be treated and constrained simultaneously.

4.1 Scalar-tensor gravity

When introducing a new gravitational scalar field, the question is: what kind of mathematical configurations can be incorporated into the Lagrangian?

There are certain principles that can be used to put constraints on the objects of a theory. These

[1]The origins of this paraphrase can be traced to a lecture given *On the Method of Theoretical Physics,* The Herbert Spencer Lecture [86, 87]

work as a sort of a reality check before even starting to compare theory and observations. One of these constraints known since the 19th century, is that greater than second order equations of motion lead to an instability [88, 89].

In this chapter we will focus on constraining the most general scalar-tensor theories. For the scalar-tensor theories with one gravitational scalar field, one can work out the most general Lagrangian that retains second order field equations. This was done in 1974 by Horndeski and the Lagrangian reads [90, 91]

$$\mathscr{L}_H = \sum_{i=2}^{5} \mathscr{L}_i, \tag{4.1}$$

where

$$\begin{aligned}
\mathscr{L}_2 &= K(\phi, X), && (4.2)\\
\mathscr{L}_3 &= -G_3(\phi, X)\Box\phi, && (4.3)\\
\mathscr{L}_4 &= G_4(\phi, X)R + G_{4X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], && (4.4)\\
\mathscr{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right] && (4.5)
\end{aligned}$$

Here the choice of function K and functions G_i correspond to a choice between different scalar-tensor theories. Here $X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ is the kinetic term for the scalar field. The derivatives

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi, \quad (\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi, \quad (\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi. \tag{4.6}$$

As mentioned in section 2.1.2, scalar-tensor theories as well as other forms of modified gravity leave room for ambiguity with regards to gravitational fields and the choice of frames. It can be shown for Horndeski theories, that if a transformation between the Jordan frame and the Einstein frame exists, it is given by the disformal transformation [92–95]. It turns out that this transformation is also the most general transformation between different frames [42, 70]. Constraining disformal transformations will be the main topic of the rest of this chapter.

4.2 Disformally coupled frames

While other frames exist (see e.g. [92]), the most common choices are the Jordan and the Einstein frames given in (2.9) and (2.14) respectively. In scalar-tensor gravity, the Jordan frame, when it exists, is the one where the scalar field couples to the Ricci scalar R and not to matter fields. Matter then follows the geodesics given by the metric tensor related to R. In the Einstein frame the field couples minimally to the curvature ($\hat{R}$) but to the matter fields instead.

In Brans-Dicke gravity [39] presented earlier in chapter 2 this change of frames is given by a conformal transformation (2.13). However, this transformation can be generalised [42, 70]. Quoting Bekenstein [42], the most general transformation between the two frames that "respects the weak equivalence principle, ordinary notions of causality, and which is insensitive to a change of zero for the auxiliary scalar field" is the so called *disformal transformation*

$$\hat{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_{,\mu}\phi_{,\nu}. \tag{4.7}$$

Here the additional scalar field is denoted by ϕ and X is its kinetic term as before.

In addition to Horndeski theories, disformal coupling comes naturally also for higher-dimensional theories with moving branes [96, 97]. They are also needed in the tensor-vector-scalar theories [73]. Thus, the disformal transformation seems to be a common feature of several different gravitational models.

In appendix A.2 we present a list of relations between the two frames.

4.3 Applications

Due to the general nature of the disformal coupling and its participating in many of the theories under Horndeski class, there is a wide variety of phenomena it has been studied in.

Although first introduced in 1993, the disformal transformation took some time to catch the attention of cosmologists [98–100]. Since then, and especially since around 2010, its effects for various cosmological and astrophysical systems have been actively studied.

Its effect has been studied at least in relation to inflation [74, 98, 101], to varying speed of light [99], interacting dark matter [102, 103], and cosmic microwave background [104, 105].

In [62, 106] the disformal coupling is considered related to the Solar system and PPN expansion. The authors conclude that constraints on the PPN parameters are probably too tight for the disformal coupling to be "relevant for cosmology". We will go through the PPN result in more detail in the following section.

However, there has also been attempts to find mechanisms to screen the effects inside dense regions, such as the Solar system [102, 107]. Some screening, or other mechanism, is needed in order for the disformally coupled scalar-tensor theories to remain a viable alternative for the dark energy. The possibility of the disformal screening is a question that is actively studied.

Some work has also been carried out considering the strong field regime. Our article [3] and the following [72] consider black holes and neutron stars respectively. Both show that the disformal coupling can lead to interesting phenomena in strong gravity fields. As [31] points out, to be able to properly study for instance gravitational radiation in modified gravity, more work needs to be done especially on the strong gravity phenomena. The LIGO observations were compatible with general relativity, but as there are no templates for merger signals in alternative theories, not much can be said regarding modified gravity.

4.4 Constraints on disformal couplings

Because the disformal transformation is the way to move from Jordan to Einstein frame in Horndeski theories, constraining the disformal coupling directly constrains properties of these theories.

So far the most stringent constrains reported come from collider experiments [108]. The reasoning goes that if standard model particles couple to the derivatives of the scalar field, then the effects would be visible in particle physics experiments. In [108] the authors used the Large Hadron Collider run 1 data to study coupling of the form

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{M^4}\phi_{,\mu}\phi_{,\nu}. \tag{4.8}$$

Here M is a constant with dimensions of mass. They find a limit of $M \gtrsim 650$ GeV. This is an orders of magnitude stronger limit than the one given by Solar system constraints, $M \gtrsim 100$ eV, which by itself was enough to render disformal coupling "irrelevant for cosmology" [62]. These two constraints suggest that to retain cosmologically interesting disformal theories we should allow for a non-universal or non-constant coupling $D(\phi, X)$.

In our article [3] we study a system where matter surrounds a black hole. There we derive a rough limit that depends on the background density as well as the disformal coupling strength. However, this limit assumes the so-called no-hair theorem, which states that the scalar field should not form a steady configuration around the black hole. This limit is somewhat more speculative, for one because as of yet we have no direct observations of such systems to test our results. Moreover, the no-hair theorem is proven for GR but remains an assumption for modified gravity theories. Hence, our limit is more speculative than the two listed above.

4.4.1 PPN parameters

A procedure similar to that presented in chapter [1] can be used to calculate the post-Newtonian parameters for disformally coupled theories [62]. Again, the metric is expanded around a flat background. For a transformation of the form

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{\Lambda^2}\phi_{,\mu}\phi_{,\nu} \tag{4.9}$$

Table 4.1: Constraints on disformal coupling from particle physics, astrophysical and cosmological systems.

Model	Limit	Effect	Reference
$\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{\Lambda^2}\phi_{,\mu}\phi_{,\nu}$	$\mathcal{M} = (M_{\mathrm{pl}}\Lambda)^{1/2} \gtrsim 100\,\mathrm{eV}$	Preferred frame effects, PPN parameter α_2	[62]
$\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{2}{M^4}\phi_{,\mu}\phi_{,\nu}$	$M \gtrsim 650\,\mathrm{GeV}$	Accelerator phyiscs	[108]
$\hat{g}_{\mu\nu} = C_0 g_{\mu\nu} + D_0\phi_{,\mu}\phi_{,\nu}$	$\frac{D_0}{C_0}\rho_0 \gtrsim 8\times 10^4$	Spontaneous scalarisation of black holes	[3]

we obtain the post-Newtonian metric

$$\begin{aligned} g_{00} &= -1 + 2U - 2N^2U^2 + 4N\Phi_1 + 4N\Phi_2 + 2\Phi_3 + 6N^2\Phi_4 + 3\Upsilon w^2 U + \Upsilon w^i w^j U_{ij} + 4\Upsilon w^i V_i, \\ g_{0i} &= -\frac{7}{2}N^2 V_i - \frac{1}{2}N^2 W_i + \Upsilon w_i U + \Upsilon w^j U_{ij}, \\ g_{ij} &= (1+2N^2U)\delta_{ij}. \end{aligned} \tag{4.10}$$

Here N is the 00 component of the metric, the so called lapse function. The definitions of the post-Newtonian potentials $(U, U_{ij}, \Phi_i, V_i, W_i)$ are listed in the appendix A.1. The standard PPN parameters presented in section 2.2.2 can be read off as

$$\gamma = \beta = 1 - \Upsilon, \tag{4.11}$$

$$\alpha_1 = -4\Upsilon, \tag{4.12}$$

$$\alpha_2 = -\Upsilon. \tag{4.13}$$

Comparing the results with the observations [18] listed in table 2.1 gives a limit to the disformal coupling strength. Authors of [62] quote it as

$$\mathcal{M} = (M_{pl}\Lambda)^{\frac{1}{2}} \gtrsim 100\,\mathrm{eV}. \tag{4.14}$$

This limit is obtained from the preferred frame effect parameter α_2.

The authors of [62] conclude that the limit forces the effects of the disformal coupling on cosmological scales to be too weak to be important. They also claim that this result can be generalised to other than constant value disformal couplings. This suggests that some sort of a mechanism is needed to screen the disformal effects in dense regions, such as the Solar system. Otherwise, the disformally coupled theories cannot remain an interesting alternative to GR.

Chapter 5

Bimetric variational principle

In this chapter we will take the idea of the independent connection of the Palatini formalism a step further. Namely, we relax the assumption of the Weyl rescaling between the physical metric and the connection generating tensor field. In these theories the two tensor fields are the fundamental degrees of freedom. The field equations are obtained by varying the action with respect to the two metrics. Hence, these models are known as "gravity models within the framework of bimetric variational formalism" [2].

Later in this chapter we present other forms of bimetric gravity theories, which have a long history in theoretical physics. Especially since 2010 massive gravity and bimetric theories have been one of the most active fields of theoretical gravity research, due to the formulation of a ghost-free massive gravity model by de Rham, Gabadadze, and Tolley [109, 110]. The bimetric variational formalism has little in common with these bimetric theories, although the problems we find it suffers from in [2] are similar to the problems that have bothered bimetric theories in general.

5.1 Story of bimetric variational formalism

Even if the spacetime metric and the connection are the fundamental degrees of freedom, intuitively it would feel natural to write the theory in terms of tensor variables. However, the spacetime connection is not a tensor, as can easily be seen by writing down the definition of the Christoffel symbol and performing a coordinate transformation. This connection is constructed of the partial derivatives of the metric tensor and the resulting composition does not inherit the tensor nature of the metric.

To fix this nuisance the first thing that comes to mind is to ask whether the independent connection could be formulated using some tensor variable. This new tensor would then give the fundamental degrees of freedom together with the metric. The equations of motion would be given by varying the action with respect to the physical metric and the new tensor. As already shown in section 3.2 in the Palatini $f(\hat{R})$ theories the independent connection can be formulated using an additional metric conformally related to the original metric tensor as

$$\hat{g}_{\mu\nu} = f'(\hat{R}) g_{\mu\nu}. \tag{5.1}$$

Then, the spacetime connection is the Christoffel symbol of this "conformal" metric.

One step towards a more general setup was taken in [111]. In this paper the authors consider a conformal transformation between the physical and the connection giving metric where the transformation is given by a function $\mathcal{C}(\hat{R})$ of the curvature scalar, namely

$$\hat{g}_{\mu\nu} = \mathcal{C}(\hat{R}) g_{\mu\nu}. \tag{5.2}$$

Here the conformal factor is some arbitrary function of the scalar curvature $\hat{R}(g, \hat{\Gamma})$.

The difference of these theories from Palatini $f(\hat{R})$ is of course in relaxing the assumption about the conformal factor. In the Palatini $f(\hat{R})$ theories the factor is the derivative with respect to the scalar curvature of the function $f(\hat{R})$ and in metric $f(R)$ the function is unity. In these theories, which the authors of [111] call C-theories, such an assumption is relaxed. Thus for example, they encompass theories in which the conformal function interpolates between the two approaches, such as

$$\mathcal{C}_\alpha(\hat{R}) = 1 + \alpha f'(\hat{R}) - \alpha, \quad \alpha \in [0, 1]. \tag{5.3}$$

For an $f(\hat{R})$ action the lower end of the range then gives metric $f(R)$, while $\alpha = 1$ results in Palatini theory.

The next logical step is of course to relax the rest of the requirements of the connection between the physical metric and the connection generating tensor. Such a model is introduced in [112, 113]. This assumption is interesting because it introduces yet another approach to generalise $f(R)$ theories. Namely, this approach introduces new dynamical degrees of freedom, as for example we show in [2]. We then obtain the C-theories and the Palatini $f(R)$ by setting extra requirements for these new degrees of freedom.

One new property allowed by this approach is non-symmetry of the spacetime connection. That is, while the physical metric and the Levi-Civita connection generated by the metric are always symmetric[1], it is not necessarily required that $\hat{\Gamma}^{\alpha}_{[\beta\gamma]} = 0$. Hence, also the new additional tensor can be non-symmetric. This property is equivalent to saying that the theories within the bimetric variational principle include propagating torsion [114]. The propagating torsion is an immediate sign

[1] We can always write $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ using symmetric $g_{\mu\nu}$. This is true even if we would have non-symmetric tensor to begin with.

that this new approach differs from the old Palatini formalism.

This approach and the nature of the new degrees of freedom is studied further in [114]. There the authors consider these theories at a linear level, expanding around Minkowski background. First, they expand an action

$$\mathscr{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(\hat{R} + 16\pi G\mathscr{L}_m\right), \tag{5.4}$$

where $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma})$. They find that at a linear level such a theory is unstable. However, they suggest that

$$\mathscr{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(\hat{R} + \alpha R), \tag{5.5}$$

where R is the usual Ricci scalar of the physical metric, gets rid of this problem for a suitable choice of the parameter α (namely $-1 < \alpha < 0$). The authors admit however, that this result is incomplete in that nonlinear effects may play role and should be studied before drawing definitive conclusions. This is what we studied in [2] and found that indeed, even the model which is healthy at linear level, contained unhealthy degrees of freedom in the non-perturbative analysis. Moreover, we found that even more general models, such as

$$\mathscr{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}f(\hat{R}), \tag{5.6}$$

are problematic and found no way of getting rid of the ghosts.

Hence, the bimetric variational principle seems to be unviable unless extra conditions are required of the two metrics. We want to emphasise once more that the fundamental difference between Palatini $f(R)$ gravity and the theories following bimetric variational principle is in the number of degrees of freedom. As shown in chapter 3, no new dynamics is introduced in Palatini $f(R)$ theories. This is due to the fact that there we can construct the a priori independent connection using a metric which is conformally related to the physical metric. If such an assumption is not made, we are left with additional degrees of freedom from the independent tensor and these then turn out to be unhealthy.

It would be worth studying how strict constraints would be required in order to get rid of the problems. For example, are C-theories the most general theories that remain ghost free or could something more general still be found? However, this question is out of the scope of this thesis.

5.2 Short history of massive gravity and the Boulware-Deser ghost

However novel the approach described above may have been, there exists a long history of problems in theories including bimetric action or massive graviton [21, 115–118]. In particular, it was long believed that all such theories suffer from the so-called Boulware-Deser ghost [117, 119]. This problem and its history are similar to the problem we found in [2]. In this section we review the history of the Boulware-Deser ghost.

It was as early as 1930's that a linearised massive gravity was successfully constructed by Fierz and Pauli [21]. The idea was to expand the metric around Minkowski background in a similar manner as in the weak field general relativity. That is to write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{5.7}$$

where $h_{\nu\mu}$ is the deviation from the flat spacetime. Then, at linear level the action of the theory reads

$$\mathscr{S} = \int \mathrm{d}^4x \left((\cdots GR \cdots) + \alpha h^{\mu\nu} h_{\mu\nu} + \beta (\eta^{\mu\nu} h_{\mu\nu})^2 \right). \tag{5.8}$$

In [21] the authors find that the only way to have such a theory viable was to require $\alpha = -\beta$, that is to fix the interaction with a mass term such that [118]

$$\mathscr{S} = \int \mathrm{d}^4x \left((\cdots GR \cdots) - \frac{1}{8} m^2 \left(h^{\mu\nu} h_{\mu\nu} - (\eta^{\mu\nu} h_{\mu\nu})^2 \right) \right). \tag{5.9}$$

The problem is that if this choice is not made, the theory has six degrees of freedom one of which allows for negative kinetic energy. Choosing $\alpha = -\beta \equiv m^2$ eliminates this problem. Hence, it is possible to construct a linear theory without ghosts for massive gravity.

However, we also need to couple the graviton to matter and for this we need to go beyond linear theory. This is where the problems reappear. In 1972 Boulware and Deser [117] showed that choosing $\alpha = -\beta$ as in (5.8) does not guarantee the absence of negative kinetic energy modes at higher orders. On the contrary, they found that generic massive spin-2 fields suffer from this and that "(a) there are necessarily six rather than the five tensor degrees of freedom, (b) the energy has no lower bound".

It remained the consensus until well into the 21st century that all Lorenz invariant massive gravity theories suffer from these problems [119]. The ghost would appear at "huge distances", well beyond the so-called Vainshtein radius, inside which the effects are screened. Moreover, the theory cannot be trusted even "inside this region, not even at the classical level".

However, it turns out it is possible to construct a ghost-free theory for massive gravity. This was done in 2010 by de Rham, Gabadadze and Tolley [109, 110]. The authors fix the action order by order so that the problematic degrees of freedom do not appear. Later the result was generalised to bimetric theories, which generically contain a massive graviton [120, 121]. The action, now known as the dRGT action reads [122]

$$\mathscr{L} = M_{Pl}^2\sqrt{-g}\left(R + 2m^2\sum_{n=0}^{3}\beta e_n(\sqrt{g^{-1}f})\right), \tag{5.10}$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are two metrics (rank two tensors) that are required by the theory.[2] The coefficients β and the polynomial functions e_k are fixed so that the ghost problem is avoided. Their exact form is not relevant for the present discussion; they can be found for example in [122].

After the introduction of dRGT gravity both massive and bimetric gravity have rapidly become one of the most active fields of theoretical gravity research (see for example [118] and references therein).

5.3 ADM formalism

In studying the properties of bimetric variational principle and theories obeying it in [2], we use formalism introduced by Arnowitt, Deser and Misner [123], generically dubbed ADM formalism. In this section we will go through the basic properties of this formalism and list a few key results. These properties and results can be found in numerous references (for example [52, 123, 124]). In [2] we expand the relevant results of ADM formalism to theories that follow the bimetric variational principle .

The motivation for developing the ADM formalism came from the need to be able to study the dynamics of the gravitational field [123]. For this we introduce the so called lapse and shift functions, N and $\vec{N}$ respectively. These functions form the 00 and the $0i$ components of the metric in a way given below. The spatial part is given by a three-dimensional tensor, the spatial metric γ_{ij}.

The intuition behind this naming is that the lapse function N appears in the time-time component of the metric, while the shift vector $\vec{N}$ resides in the $0i$ components. Consider a slicing of spacetime where three-dimensional spatial slices are separated by time interval $\mathrm{d}t$ (Figure 5.1). The proper time between two events comoving with the slices (at points $(t, \vec{x})$ and $(t, \vec{x}) + \mathrm{d}\tau$) is given by the lapse and the shift functions, $N(t, \vec{x})$ and $\vec{N}(t, \vec{x})$ respectively, such that

$$\mathrm{d}\tau = \left(N(t, \vec{x}), \vec{N}(t, \vec{x})\right)\,\mathrm{d}t. \tag{5.11}$$

The spatial displacement $\mathrm{d}\vec{l}$ is due to a displacement (length $\mathrm{d}l^2 = \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$) on the slice given by the three-dimensional metric γ_{ij} on the spatial slice added to the displacement, or shifting, of the slices with respect to one another. This latter movement is given by the shift (vector) function

[2] Even if the theory is single metric massive gravity an additional non-dynamical tensor is needed.

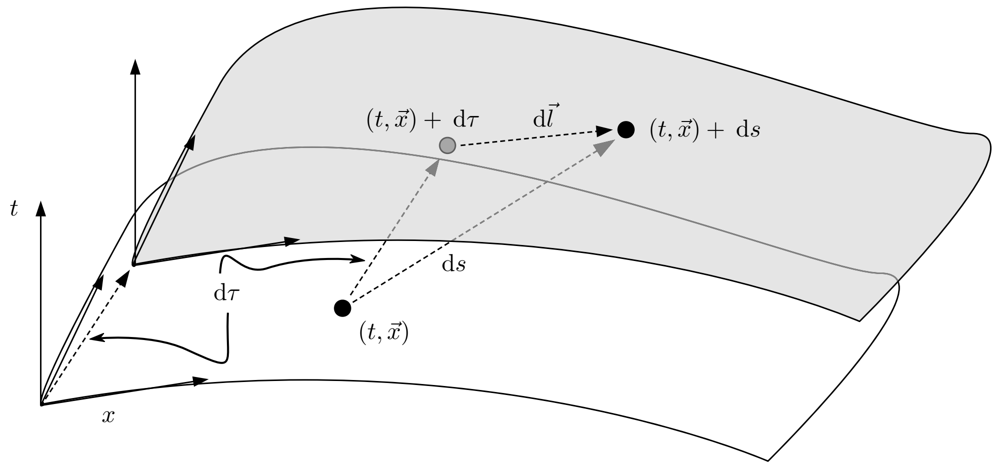


Figure 5.1: The total spacetime interval $\mathrm{d}s$ between events $(t,\vec{x})$ and $(t,\vec{x})+\mathrm{d}s$ is given by the three dimensional displacement on the sheet $\mathrm{d}\vec{l}$ added to the displacement of the sheet $\mathrm{d}\tau = \left(N, N^i\right)\,\mathrm{d}t$. The three dimensional interval length is given by the three dimensional metric γ_{ij} such that $\mathrm{d}l^2 = \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$.

$\vec{N}(t,\vec{x})\,\mathrm{d}t$. Hence, the infinitesimal interval, the proper distance, from an event $(t,\vec{x})$ to another event on a different slice $(t,\vec{x})+\,\mathrm{d}s$ is

$$\mathrm{d}s^2 = \left(N\,\mathrm{d}t, \vec{N}\,\mathrm{d}t + \,\mathrm{d}\vec{l}\right)^2 = -N^2\,\mathrm{d}t^2 + \gamma_{ij}(N^i\,\mathrm{d}t + \,\mathrm{d}x^i)(N^j\,\mathrm{d}t + \,\mathrm{d}x^j) \tag{5.12}$$

With these choices the ADM metric can be read off the line element

$$g_{\mu\nu} = \begin{pmatrix} -\left(N^2 - N_i N^i\right) & N_j \\ N_i & \gamma_{ij} \end{pmatrix}. \tag{5.13}$$

The spatial metric γ_{ij} and its contravariant counterpart are used to raise and lower spatial indices (denoted by latin alphabet). It is worth mentioning that the contravariant γ^{ij} does not alone construct the spatial part of the inverse metric. Instead, if we solve it for instance from $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$, we get

$$g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix}. \tag{5.14}$$

For applications it is useful to define a unit vector $n^\mu \equiv \left(\frac{1}{N}, -\frac{N^i}{N}\right)$. This vector definition also gives a three dimensional slicing of the spacetime. Namely, each spatial (three-dimensional) slice of this slicing is orthogonal to n^μ at each event. The corresponding covariant dual counterpart of n^μ is given by the one-form $n_\mu \equiv (-N, 0)\,(= g_{\mu\nu}n^\mu)$.

With the ADM composition all of the relevant general relativistic objects can be written [52, 123, 124]. For instance,

$$
\begin{aligned}
\sqrt{g} &= N\sqrt{\gamma} && (5.15)\\
R &= \overset{(3)}{R} + K^{ij}K_{ij} + K^i_i K^j_j - \frac{2}{N}\dot{K}^i_i + 2\frac{N^j}{N}\overset{(3)}{\nabla}_j K^i_i - \frac{2}{N}\overset{(3)}{\triangle} N, && (5.16)
\end{aligned}
$$

where we use the extrinsic curvature of a constant time slice (where $t = \text{const.}$)

$$
K_{ij} \equiv -\nabla_i n_j = \Gamma^{\mu}_{ij} n_\mu = -N\Gamma^0_{ij}. \tag{5.17}
$$

Hence the Einstein-Hilbert action reads

$$
\mathscr{S} = \int \mathrm{d}x^4 \sqrt{g} R = \int \mathrm{d}t\,\mathrm{d}x^3 \sqrt{\gamma} N \left(\overset{(3)}{R} + K^{ij}K_{ij} - K^i_i K^j_j \right). \tag{5.18}
$$

The difference between (5.16) and (5.18) is due to dropping a total time derivative and a spatial total divergence. These only contribute surface terms to the action and varying the geometry inside the surface does not alter them. Hence, they do not contribute to local variation of the action nor to the equations of motion [52].

In [2] we develop and derive the ADM Hamiltonian for a theory following bimetric variational principle. And as for historical massive gravity, while these theories can be constructed to be ghost-free at the linear level, we find that writing the full theory reintroduces the ghost problems.

In practice, in [2], we write Lagrangians (5.4),(5.5), and (5.6) in terms of the ADM variables. The problems appear clearly as we transform the Lagrangian into the Hamiltonian. Namely we end up with a dynamical field term, which is not bound from either side, especially not from below. This means that by emitting energy the field can decay to lower energy states ad infinitum, which is of course unphysical. This is the same problem as the one described in section 5.2.

Chapter 6

Conclusions

During the past hundred years general relativity has served as the standard model of gravity and cosmology. However, its shortcomings, such as non-renormalisability or the problems of dark energy, remain a subject of scientific research. Many modified gravity models address one or more of these shortcomings. There are models that are renormalisable. There are also models which do not require dark energy, or exotic additional scalar fields in order to explain the late accelerated expansion of the universe.

This thesis is based on three articles concerning modified gravity. The first part introduces briefly the subjects discussed in the three articles. The second part are the actual research articles published in refereed journals.

We begin by introducing some basic concepts of gravity theories in chapter 2. We show the basic procedure for obtaining the geodesics and field equations using Brans-Dicke scalar-tensor theory as an example. We also introduce the PPN formalism to compare different gravity theories with observations. Lastly we go through different formulations of equivalence principles and discuss how they can be used to divide theories into those that have dark energy and to those called modified gravity.

In chapter 3 present the basic ideas of the Palatini f(R) theories of gravity. We discuss Palatini theories in article [1], where we study the equations of motion of compact objects. We derive the relative acceleration of individual components of a system of compact objects. We show that for the Palatini $f(R)$ theories, independent observations of masses are needed in order to distinguish between GR and Palatini f(R) gravity. In the level of equations of motion the only effect of the modification is in the scaling of masses. In the absence of an independent mass measurement, the theories are observationally indistinguishable.

In chapter 3 we also discuss the equivalence of f(R) and scalar-tensor theories of gravity. We point out that some of the results of scalar-tensor theories in the literature diverge when the Brans-Dicke parameter $\omega \to -3/2$. This is the case for Palatini f(R) gravity and therefore, Palatini f(R) cannot be taken as the correct limit for those results [85].

Chapter 4 deals with scalar-tensor theories of gravity with the so called disformal coupling. In these theories matter couples to geometry via derivatives of the scalar field. We go through some formalism and motivation for the disformal coupling. Between frames, the disformal coupling is the most general transformation that retains weak equivalence. It is also the possible transformation between the Einstein and the Jordan frames in the Horndeski scalar-tensor theories.

In our article [3] we studied systems of black holes and surrounding matter in disformally coupled theories. We showed that the disformal coupling can amplify the spontaneous scalarisation instability. However, for suitable coupling strengths the effect can also be stabilising. This result requires further studying, especially regarding observations. In principle, our result could eventually be used to constrain disformal coupling between matter and geometry. However, as mentioned in chapter **??**, constraints from the LHC and solar system observations require screening of disformal effects in denser regions, such as matter distributions around black holes.

In chapter 5 we review the history of bimetric gravity and the bimetric variational principle. Theories following this principle can be used for example to generalise f(R) gravity to classes that contain both the metric and the Palatini formulation as special cases. In some sense theories following bimetric variational formalism can be seen as the next logical step beyond the Palatini formulation. Whereas Palatini f(R) takes the spacetime connection to be independent of the metric, the bimetric variational principle assumes that the independent connection itself is constructed from another rank two tensor. The two tensors then yield the fundamental degrees of freedom in these theories.

We also give a short introduction to the ADM formalism. It is a way to parametrise the spacetime to study its dynamics. We use the ADM formalism in [2] where we show that the problems of general bimetric gravity are present in the theories following the bimetric variational formalism. Exactly as for the historical bimetric theories, these theories avoid problems when linearised around a flat background. However, again exactly as in the historical bimetric theories problems reappear twice as complicated in the exact Hamiltonian.

More than anything, the number and the diversity of models of gravity is a consequence of the weakness of gravitational interaction leaving room for speculation. Making direct observations of phenomena beyond weak fields has been impossible prior to 2015 when a collision of two black holes was detected by the LIGO gravitational wave detectors. The non-existence of direct observations allows for a wide variety of possibilities to remain, well, possible until constrained by future measurements.

In the coming two decades gravity research will make at least two major observational leaps forward. The first is the growing number of gravitational wave detectors that will allow us to observe the sky over a variety of wave lengths. Combined these detectors will yield a whole new observational window for astronomers to compare gravity models. The second big step will be the ESA Euclid satellite, which will measure the dark matter distribution in the universe. Euclid will also be important in determining the nature of dark energy, or effects of modified gravity, through the time evolution of the dark matter distribution.

Future observational improvements call for gravity theories that have predictive power and can acts as a guide for observational strategies. Both of the above mentioned forward leaps will allow us to enhance our knowledge of the universe and improve our models of the weakest of the fundamental interactions.

Appendix A

Definitions and intermediate results

A.1 Newtonian and post-Newtonian potentials

In the considerations of post-Newtonian equations of motion the following notation is used for the gravitational potentials. The notation follows that of [18]

$$
\begin{aligned}
U &= \int \frac{\rho'}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
U_{ij} &= \int \frac{\rho'(x-x')_i(x-x')_j}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}^3x', \\
\Phi_W &= \int \frac{\rho'\rho''(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\cdot\left(\frac{\mathbf{x}'-\mathbf{x}''}{|\mathbf{x}-\mathbf{x}''|}-\frac{\mathbf{x}-\mathbf{x}''}{|\mathbf{x}'-\mathbf{x}''|}\right)\mathrm{d}^3x'\,\mathrm{d}^3x'', \\
\mathcal{A} &= \int \frac{\rho'\left[\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')\right]^2}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}^3x', \\
\Phi_1 &= \int \frac{\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
\Phi_2 &= \int \frac{\rho' U'}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
\Phi_3 &= \int \frac{\rho' \Pi'}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
\Phi_4 &= \int \frac{p'}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
V_i &= \int \frac{\rho' v'_i}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x', \\
W_i &= \int \frac{\rho'\left[\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')\right](x-x')_i}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x'.
\end{aligned}
$$

A.2 Disformal relations

First of all, the tensor indices of two disformally related frames are raised and lowered with the metric of the corresponding frame. Thus, for example [92]

$$\hat{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_{,\mu}\phi_{,\nu} \tag{A.1}$$

$$\hat{g}^{\mu\nu} = \frac{1}{C(\phi, X)}\left(g^{\mu\nu} - \epsilon^2\phi^{,\mu}\phi^{,\nu}\right) \tag{A.2}$$

where

$$\epsilon^2 \equiv \frac{D(\phi, X)}{C(\phi, X) - 2D(\phi, X)X}. \tag{A.3}$$

It should be noted that the transformation from one frame to another affects $C(\phi, X)$ and $D(\phi, X)$ via the kinetic term. Therefore, to write the inverse relations using exclusively Einstein frame variables may not be a straight forward task. However, for transformation functions that do not depend on the field kinetics the relations may be written as

$$g_{\mu\nu} = \frac{1}{C(\phi)}\left(\hat{g}_{\mu\nu} - D(\phi)\phi_{,\mu}\phi_{,\nu}\right), \tag{A.4}$$

$$g^{\mu\nu} = C(\phi)\hat{g}^{\mu\nu} + \hat{\epsilon}^2\hat{\phi}^{,\mu}\hat{\phi}^{,\nu}. \tag{A.5}$$

where the right hand side consists exclusively on Einstein frame variables. The coefficient corresponding to ϵ above is

$$\hat{\epsilon}^2 = \frac{D(\phi)}{1 + 2D(\phi)\hat{X}}. \tag{A.6}$$

In addition, following the notation of [1] we will sometimes use the Lorentz-factor

$$\gamma^2 \equiv \frac{C(\phi, X)}{C(\phi, X) - 2D(\phi, X)X}. \tag{A.7}$$

In the Einstein frame the corresponding factor is

$$\hat{\gamma}^2 \equiv \frac{C(\phi, \hat{X})}{1 + 2D(\phi, \hat{X})\hat{X}}. \tag{A.8}$$

With these coefficients the covariant vector transforms as $\phi^{,\mu} = \hat{\gamma}^2\hat{\phi}^{,\mu}$.[1]

[1] The different coefficients are related as

$$\frac{1}{C}\gamma^2 = \frac{1}{D}\epsilon^2 = \frac{D}{C}\hat{\epsilon}^{-2} = \hat{\gamma}^{-2} \tag{A.9}$$

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